III Geometric group theory – Example Sheet 4

Lent 2024 `hjrw2@cam.ac.uk`

1. Consider the infinite path $\gamma : [1, \infty) \to \mathbb{R}^2$ defined by
$$\gamma(t) = (t\cos(\log t), t\sin(\log t)) .$$
Prove that γ is a quasigeodesic (for suitable constants). Deduce that the analogue of the Mostow–Morse lemma fails for $\mathbb{R}^2$. [*You may use that the length of a smooth path in* $\mathbb{R}^2$ *can be computed using the integral* $\int \|\gamma'\| dt$.]

2. Two groups G_1 and G_2 are called *(abstractly) commensurable* if there are subgroups $H_i \leq G_i$ such that $|G_i : H_i| < \infty$ and $H_1 \cong H_2$. Prove that commensurable finitely generated groups are quasi-isometric.

3. Let P be a geodesic n-gon in a δ-hyperbolic metric space X. Prove that every side of P is contained in the closed $(n-2)\delta$-neighbourhood of the other sides of P.

4. Let ϕ be an isometry of a δ-hyperbolic metric space X with no fixed point. Suppose that $\alpha, \beta : \mathbb{R} \to X$ are both geodesic lines preserved by ϕ. Prove that $d_{\text{Haus}}(\operatorname{im}\alpha, \operatorname{im}\beta)$ is finite, and furthermore that $d_{\text{Haus}}(\operatorname{im}\alpha, \operatorname{im}\beta) \leq 2\delta$.

5. Let ϕ be an isometry of a metric space X.
   (a) For any $x \in X$, prove that the sequence of real numbers
   $$t_n = \frac{d(x, g^n x)}{n}$$
   is convergent. [*You may use without proof Fekete's subadditivity lemma. A sequence* a_n *is called* subadditive *if* $a_{m+n} \leq a_m + a_n$ *for all* m, n. *Fekete's lemma asserts that* a_n/n *converges if* a_n *is subadditive.*]
   (b) Prove that $\tau(\phi) = \lim_n t_n$ does not depend on the choice of x.

   The quantity $\tau(\phi)$ is called the *translation length* of ϕ.

6. Let X be a δ-hyperbolic metric space, and let $\alpha, \beta : [0, L] \to X$ be geodesics with $\alpha(0) = \beta(0)$. Prove that
$$d(\alpha(t), \beta(t)) \leq 2\delta + d(\alpha(L), \beta(L))$$
for all $t \in [0, L]$.

7. Let X be a proper metric space (i.e. closed balls are compact). A subspace $Y \subseteq X$ is called *convex* if every geodesic in X with endpoints in Y is contained in Y.
   (a) For any $x \in X$ and any non-empty closed subspace $Y \subseteq X$, prove that there is $y_0 \in Y$ such that $d(x, y_0) \leq d(x, y)$ for all $y \in Y$.
   (b) Give an example of a δ-hyperbolic metric space X, a closed, convex subset $Y \subseteq X$, a point $x \in X$ and a pair of *distinct* points $y_1, y_2 \in Y$ that both minimise distance to x among all points in Y.
   (c) Let X be δ-hyperbolic and Y a convex subspace. Suppose that $y_1, y_2 \in Y$ both have the property that $d(x, y_i) \leq d(x, y)$ for all $y \in Y$. Prove that $d(y_1, y_2) \leq 4\delta$.

8. Let X be a geodesic metric space and consider a geodesic triangle $\Delta = [x,y] \cup [y,z] \cup [z,x]$ in X.

   (a) Let $p \in [x,y]$ be the point such that
   $$2d(p,x) = d(y,x) + d(z,x) - d(y,z)$$
   and let $p' \in [x,y]$ be such that
   $$2d(p',y) = d(x,y) + d(z,y) - d(x,z)\,.$$
   Prove that $p = p'$.

   (b) Define $q \in [x,z]$ similarly to $p \in [x,y]$. Prove that $d(x,p) = d(x,q)$ and $d(p,q) \leq 4\delta$.

   (c) For any $a \in [x,y]$ and $b \in [x,z]$ with
   $$d(x,a) = d(x,b) \leq d(x,p)\,,$$
   prove that $d(a,b) \leq 6\delta$.

9. Let X be a δ-hyperbolic space and $Y \subseteq X$ a bounded subspace. Consider the function $R_Y : X \to \mathbb{R}_{\geq 0}$ defined by
   $$R_Y(x) = \sup_{y \in Y} d(x,y)\,.$$
   The *radius* of Y is defined to be
   $$r_Y := \inf_{x \in X} R_Y(x)\,.$$
   A point $x \in X$ is called an ϵ-*centre* of Y if $R_Y(x) \leq r_Y + \epsilon$. Prove that, if x, x' are both ϵ-centres of Y, then $d(x,x') \leq 12\delta + 2\epsilon$. [*Hint: Let m be the midpoint of $[x,x']$ and consider $y \in Y$ such that $d(m,y) \geq r_Y$.*]

10. Let G be a hyperbolic group and S a finite generating set. Let $\gamma \in G$ be an element of finite order.

    (a) Show that $\langle \gamma \rangle \subseteq \mathrm{Cay}_S(G)$ has a 1-centre $g \in G$, in the sense of Question 9.

    (b) Prove that γg is also a 1-centre of $\langle \gamma \rangle$.

    (c) Deduce that G has finitely many conjugacy classes of elements of finite order.